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OPERATIONAL METHOD IN FRACTIONAL CALCULUS

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Abstract

Operational calculi for various differential operators of hyper-Bessel type have been successfully used for solving some problems in different fields including ordinary and partial differential equations, integral equations and theory of special functions. Recently Mikusiński's scheme has been applied to develop operational calculi for some basic operators of fractional calculus, namely, for the Riemann-Liouville fractional derivative, for the Caputo fractional derivative and for the more general multiple Erdélyi-Kober fractional derivative.

In this survey paper we give some elements of these operational calculi and some of their applications with special emphasis of the operational calculus for the Caputo fractional derivative and operational method for solving initial value problems for the general n-term linear equation with the Caputo derivatives of arbitrary orders and constant coefficients. Special cases and integral representations of solutions are presented. In all the cases the obtained solutions are expressed through the Mittag-Leffler type functions.

Mathematics Subject Classification: 26A33 (main), 44A40, 44A35, 33E30, 45J05, 45D05

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1. Introduction

The fractional differential equations have excited in recent years a considerable interest both in mathematics and in applications. They have been used in modeling of many physical and chemical processes and in engineering (see, for example, [2] – [6], [13], [14], [23], [24], [30]). In its turn, mathematical aspects of fractional differential equations and methods of their solution have been discussed by many authors including [2]–[4], [13]– [17], [20]–[26], [28], [29] and [32]. In the mathematical treatises on fractional differential equations the Riemann-Liouville approach to the notion of the fractional derivative of order μ $(m-1 < \mu \le m \in \mathbf{N})$ is normally used:

$$(D^{\mu}y)(t) := \left(\frac{d}{dt}\right)^{m} (J^{m-\mu}y)(t),\ t > 0. \tag{1}$$

Here

$$(J^{\mu}y)(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} y(\tau)\, d\tau,\ \mu > 0,\ t > 0, \tag{2}$$

$$(J^{0}y)(t) := y(t),\ t > 0$$

is the Riemann-Liouville fractional integral of order μ. In this interpretation, the fractional derivative is left-inverse (and not right-inverse) of the fractional integral, which is a natural generalization of the Cauchy formula for the n-fold primitive of a function y. As to the initial value problems for fractional differential equations with fractional derivatives in the Riemann-Liouville sense, there are some troubles with the initial conditions, see [15], [21], [28], [29]– [30]. Namely, these initial conditions should be given as (bounded) initial values of the fractional integral $J^{m-\mu}y$ and of its integer derivatives of order $k = 1, 2, \dots, m-1$. On the other hand, in modeling of real processes, the initial conditions are normally expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order. In order to meet this physical requirement, an alternative definition of fractional derivative was introduced by Caputo [5] and adopted by Caputo and Mainardi [6] in the framework of the theory of linear viscoelasticity:

$$(D_*^{\mu}y)(t) := (J^{m-\mu}y^{(m)})(t), \quad m-1 < \mu \le m \in \mathbf{N},\ t > 0. \tag{3}$$

In this survey paper based on the original works of the author and his co-authors, some elements of the operational calculi for the Riemann-Liouville, the Caputo and the more general multiple Erdélyi-Kober fractional derivatives will be given with a special emphasis of an operational method for solving differential equations of fractional order. In our discussions we follow the results presented by Gorenflo and Luchko [11], Hadid and Luchko [15] and Luchko and Srivastava [21] in the case of the Riemann-Liouville fractional derivative, by Luchko and Gorenflo [20] in the case of the Caputo fractional derivative and by Al-Bassam and Luchko [1] and Luchko and Yakubovich [22] in the case of the multiple Erdélyi-Kober fractional derivative. The operational method for solving integral equations of Abel's type was given by Gorenflo and Luchko [11] and Gorenflo, Luchko and Srivastava [12]. In all the cases we omit the proofs which can be found in the corresponding original papers.

2. Operational calculi for fractional derivatives

In this section we first develop an operational calculus for the Caputo fractional derivative and then give some elements of the operational calculi for the Riemann-Liouville and for the multiple Erdélyi-Kober fractional derivatives.

We begin by defining the function space $C_\alpha,\ \alpha \in \mathbf{R}$, which was introduced for the first time by Dimovski in his papers devoted to the operational calculus for hyper-Bessel differential operators (see e.g. [7],[9]).

DEFINITION 2.1. A real or complex-valued function y, is said to be in the space $C_\alpha,\ \alpha \in \mathbf{R}$, if there exists a real number p, $p > \alpha$, such that

$$y(t) = t^p y_1(t),\ t > 0$$

with a function $y_1 \in C[0,\infty)$.

Clearly, C_α is a vector space and the set of spaces C_α is ordered by inclusion according to

$$C_\alpha \subset C_\beta \Leftrightarrow \alpha \ge \beta. \tag{4}$$

THEOREM 2.1. *The Riemann-Liouville fractional integral $J^\mu,\ \mu \ge 0$, is a linear map of the space $C_\alpha,\ \alpha \ge -1$, into itself, that is,*

$$J^\mu : C_\alpha \to C_{\mu+\alpha} \subset C_\alpha.$$

REMARK 2.1. *In the case $y \in C_\alpha$ for a value $\alpha \ge -1$ and for $\mu \ge 1$ we have $J^\mu y \in C_0 \subset C[0,\infty)$.*

It is important to note, that the operator J^μ, $\mu > 0$ has the following convolution representation in the space C_α, $\alpha \ge -1$:

$$(J^\mu y)(t) = (h_\mu \circ y)(t),\ h_\mu(t) := t^{\mu-1}/\Gamma(\mu), \quad y \in C_\alpha. \tag{5}$$

Here

$$(g \circ f)(t) = \int_0^t g(t-\tau)f(\tau)\,d\tau,\ t > 0$$

is the Laplace convolution. For the Laplace convolution itself, the inclusion

$$g \circ f \in C_{\alpha_1+\alpha_2+1} \subseteq C_{-1}, \quad f \in C_{\alpha_1},\ g \in C_{\alpha_2},\ \alpha_1, \alpha_2 \ge -1 \tag{6}$$

holds true. Representation (5) and the commutativity of the Laplace convolution (see [9],[27]) lead to the following property of the Riemann-Liouville fractional integral:

$$(J^\delta J^\eta y)(t) = (J^\eta J^\delta y)(t),\ y \in C_\alpha,\ \alpha \ge -1,\ \delta \ge 0,\ \eta \ge 0.$$

Next, using the associativity of the Laplace convolution and the Euler integral of the first kind for the evaluation of $(h_\delta \circ h_\eta)(t)$, we obtain

$$(J^\delta J^\eta y)(t) = (J^{\delta+\eta} y)(t),\ y \in C_\alpha,\ \alpha \ge -1,\ \delta \ge 0,\ \eta \ge 0, \tag{7}$$

that is also well known. In particular, it follows from (7) that

$$(\underbrace{J^\mu \dots J^\mu}_{n} y)(t) = (J^{n\mu} y)(t),\ y \in C_\alpha,\ \alpha \ge -1,\ \mu \ge 0,\ n \in \mathbf{N}. \tag{8}$$

It is obvious, that the Caputo fractional derivative (3) is not defined on the whole space C_α. Let us introduce a subspace of C_α, which is suitable for dealing with the Caputo derivative.

Definition 2.2. A function y is said to be in the space $C_\alpha^m,\ m \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, if $y^{(m)} \in C_\alpha$.

Remark 2.2. *The space* C_α^m *does not coincide with the space* $C_\alpha^{(m)} = \{y : y(t) = t^p \tilde{y}(t),\ t > 0,\ p > \alpha,\ \tilde{y} \in C^m[0,\infty)\}$, *considered in [18]. For example, if* $y(t) = \cos(t)/\sqrt{t},\ t > 0$, *then* $y \in C_{-1}^{(1)},\ y \notin C_{-1}^1$ *and for the function* $y(t) \equiv 1,\ t > 0$ *we have* $y \notin C_0^{(1)},\ y \in C_0^1$.

We give some properties of the space C_α^m:

1) C_α^m is a vector space.

2) $C_\alpha^0 \equiv C_\alpha$.

3) If $y \in C_\alpha^m$ for a value $\alpha \ge -1$ and an index $m \ge 1$, then $y^{(k)}(0+) := \lim_{t \to 0+} y^{(k)}(t) < +\infty,\ 0 \le k \le m-1$ and the function

$$\tilde{y}(t) = \begin{cases} y(t), & t > 0, \\ y(0+), & t = 0 \end{cases}$$

is in $C^{m-1}[0,\infty)$.

4) If $y \in C_\alpha^m$ for a value $\alpha \ge -1$, then $y \in C^m(0,\infty) \cap C^{m-1}[0,\infty)$.

5) For real $\alpha \ge -1$ and index $m \ge 1$ the following equivalence holds:

$$y \in C_\alpha^m \Leftrightarrow y(t) = (J^m \phi)(t) + \sum_{k=0}^{m-1} c_k \frac{t^k}{k!},\ t \ge 0,\ \phi \in C_\alpha.$$

On the basis of the properties 1)-5) of the functional space C_α^m we obtain some theorems, important for the development of the corresponding operational calculus.

THEOREM 2.2. *Let $y \in C_{-1}^m,\ m \in \mathbf{N}_0$. Then the Caputo fractional derivative $D_*^\mu y,\ 0 \le \mu \le m$ is well defined and the inclusion*

$$D_*^\mu y \in \begin{cases} C_{-1}, & m-1 < \mu \le m, \\ C^{k-1}[0,\infty) \subset C_{-1}, & m-k-1 < \mu \le m-k,\ k = 1, \ldots, m-1 \end{cases}$$

holds true.

THEOREM 2.3. *Let $y \in C_{-1}^m,\ m \in \mathbf{N}$ and $m-1 < \mu \le m$. Then the Riemann-Liouville and the Caputo fractional derivatives are connected by the relation:*

$$(D^\mu y)(x) = (D_*^\mu y)(x) + \sum_{k=0}^{m-1} \frac{y^{(k)}(0+)}{\Gamma(1+k-\mu)} t^{k-\mu},\ t > 0. \tag{9}$$

REMARK 2.3. *It follows from representation (9) that the Riemann-Liouville fractional derivative $D^\mu y$ is not, in the general case, in the space C_{-1}, if $y \in C_{-1}^m$. There are only three exceptional cases:*

1) If $\mu = m \in \mathbf{N}$, then

$$D^\mu y \equiv D_*^\mu y \equiv y^{(m)}\ \in C_{-1}.$$

2) If $y^{(k)}(0+) = 0,\ k = 0, \ldots, m-1$, then

$$D^\mu y \equiv D_*^\mu y\ \in C_{-1}.$$

3) If $0 < \mu < 1$, then $D^\mu y \in C_{-1}$ because of

$$(D^\mu y)(t) = (D_*^\mu y)(t) + \frac{y(0+)}{\Gamma(1-\mu)} t^{-\mu}.$$

THEOREM 2.4. *Let $m-1 < \mu \le m,\ m \in \mathbf{N},\ \alpha \ge -1$ and $y \in C_\alpha^m$. Then*

$$(J^\mu D_*^\mu y)(t) = y(t) - \sum_{k=0}^{m-1} y^{(k)}(0+) \frac{t^k}{k!},\ t \ge 0. \tag{10}$$

THEOREM 2.5. *Let $f \in C_{-1}^m,\ m \in \mathbf{N}_0,\ f(0) = \ldots = f^{(m-1)}(0) = 0$ and $g \in C_{-1}^1$. Then the Laplace convolution*

$$h(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau$$

is in the space C_{-1}^{m+1} and $h(0) = \ldots = h^{(m)}(0) = 0$.

For the sake of simplicity we shall consider in our further discussions the case of the space C_{-1}, which turns out to be most interesting one for applications. As in the case of Mikusiński's calculus, we have the following theorem.

THEOREM 2.6. *The space C_{-1} with the operations of the Laplace convolution $\circ$ and ordinary addition becomes a commutative ring $(C_{-1}, \circ, +)$ without divisors of zero.*

This ring can be extended to the field $\mathcal{M}_{-1}$ of convolution quotients by following the lines of Mikusiński [27]:

$$\mathcal{M}_{-1} := C_{-1} \times (C_{-1} \setminus \{0\}) / \sim,$$

where the equivalence relation $(\sim)$ is defined, as usual, by

$$(f, g) \sim (f_1, g_1) \Leftrightarrow (f \circ g_1)(t) = (g \circ f_1)(t).$$

Thus, we can consider the elements of the field $\mathcal{M}_{-1}$ as convolution quotients f/g and define the operations in $\mathcal{M}_{-1}$ as follows:

$$\frac{f}{g} + \frac{f_1}{g_1} := \frac{f \circ g_1 + g \circ f_1}{g \circ g_1}$$

and

$$\frac{f}{g} \cdot \frac{f_1}{g_1} := \frac{f \circ f_1}{g \circ g_1}.$$

The proof of the fact that the set $\mathcal{M}_{-1}$ is a commutative field with respect to operations "+" and "$\cdot$" is based on Theorem 2.6.

It is easily seen that the ring C_{-1} can be embedded into the field $\mathcal{M}_{-1}$ by the map $(\mu > 0)$:

$$f \mapsto \frac{h_\mu \circ f}{h_\mu},$$

with, by (5), $h_\mu(t) = t^{\mu-1}/\Gamma(\mu)$.

Defining the operation of multiplication with a scalar λ from the field **R** (or **C**) by the relation

$$\lambda \frac{f}{g} := \frac{\lambda f}{g}, \ \frac{f}{g} \in \mathcal{M}_{-1}$$

and remembering the fact, that the set C_{-1} is a vector space, we check that the set $\mathcal{M}_{-1}$ is a vector space too. Since the constant function $f(t) \equiv \lambda, \ t > 0$ is in the space C_{-1}, we should distinguish the operation of multiplication with a scalar in the vector space $\mathcal{M}_{-1}$ and the operation of multiplication with a constant function in the field $\mathcal{M}_{-1}$. In this last case we shall write

$$\{\lambda\} \cdot \frac{f}{g} = \frac{\lambda h_{\mu+1}}{h_\mu} \cdot \frac{f}{g} = \{1\} \cdot \frac{\lambda f}{g}. \tag{11}$$

It can easily be checked that the element $I = \frac{h_\mu}{h_\mu}$ of the field $\mathcal{M}_{-1}$ is the unity of this field with respect to the operation of multiplication. From the other side this element of the field $\mathcal{M}_{-1}$ is not reduced to a function from the ring C_{-1} and, consequently, it can be regarded as a generalized function. Later we shall consider some other elements of the field $\mathcal{M}_{-1}$ possessing this property, in particular, the element which will play an important role in the applications of operational calculus and is given by

Definition 2.3. The algebraic inverse of the Riemann-Liouville fractional operator J^μ is said to be the element S_μ of the field $\mathcal{M}_{-1}$, which is reciprocal to the element h_μ in the field $\mathcal{M}_{-1}$, that is,

$$S_\mu = \frac{I}{h_\mu} \equiv \frac{h_\mu}{h_\mu \circ h_\mu} \equiv \frac{h_\mu}{h_{2\mu}}\,, \tag{12}$$

where (and in what follows) $I = \frac{h_\mu}{h_\mu}$ denotes the identity element of the field $\mathcal{M}_{-1}$ with respect to the operation of multiplication.

As we have already seen, the Riemann-Liouville fractional integral J^μ can be represented as a multiplication (convolution) in the ring C_{-1} (with the function h_μ, see (5)). Since the ring C_{-1} is embedded into the field $\mathcal{M}_{-1}$ of convolution quotients, this fact can be rewritten as follows:

$$(J^\mu y)(t) = \frac{I}{S_\mu} \cdot y. \tag{13}$$

As to the Caputo fractional derivative, there exists no convolution representation in the ring C_{-1} for it, but it is reduced to the operator of multiplication in the field $\mathcal{M}_{-1}$.

Theorem 2.7. *Let $f \in C^m_{-1},\ m-1 < \mu \le m,\ m \in \mathbf{N}$. Then the following relation holds true in the field $\mathcal{M}_{-1}$ of convolution quotients:*

$$D^\mu_* y = S_\mu \cdot y - S_\mu \cdot y_\mu,\ y_\mu(t) := \sum_{k=0}^{m-1} y^{(k)}(0+)\frac{t^k}{k!}\,. \tag{14}$$

We already know (see (8)), that for $\mu > 0,\ n \in \mathbf{N}$

$$h^n_\mu(t) := \underbrace{h_\mu \circ \ldots \circ h_\mu}_{n} = h_{n\mu}(t).$$

Let us extend this relation to an arbitrary positive real power exponent:

$$h^\lambda_\mu(t) := h_{\lambda\mu}(t),\ \lambda > 0. \tag{15}$$

We have for any $\lambda > 0$ the inclusion $h^\lambda_\mu \in C_{-1}$, and the following relations can be easily checked $(\alpha > 0,\ \beta > 0)$:

$$h^\alpha_\mu \circ h^\beta_\mu = h_{\alpha\mu} \circ h_{\beta\mu} = h_{(\alpha+\beta)\mu} = h^{\alpha+\beta}_\mu, \tag{16}$$

$$h_{\mu_1}^{\alpha} = h_{\mu_2}^{\beta} \;\Leftrightarrow\; \mu_1\alpha = \mu_2\beta. \tag{17}$$

Then we define a power function of the element S_μ with an arbitrary real power exponent λ:

$$S_\mu^\lambda = \begin{cases} h_\mu^{-\lambda}, & \lambda < 0, \\ I, & \lambda = 0, \\ \frac{I}{h_\mu^\lambda}, & \lambda > 0. \end{cases} \tag{18}$$

Using this definition and the relations (16) and (17), we get ($\alpha, \beta \in \mathbf{R}$):

$$S_\mu^\alpha \cdot S_\mu^\beta = S_\mu^{\alpha+\beta}, \tag{19}$$

$$S_{\mu_1}^{\alpha} = S_{\mu_2}^{\beta} \;\Leftrightarrow\; \mu_1\alpha = \mu_2\beta. \tag{20}$$

For many applications it is important to know the functions of S_μ in $\mathcal{M}_{-1}$ which can be represented by means of the elements of the ring C_{-1}. One useful class of such functions is given by the following theorem.

THEOREM 2.8. *Let the multiple power series*

$$\sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} z_1^{i_1} \times \cdots \times z_n^{i_n}, \quad z_1,\ldots,z_n \in \mathbf{C},\; a_{i_1,\ldots,i_n} \in \mathbf{C}$$

be convergent at a point $z_0 = (z_{10},\ldots,z_{n0})$ *with all* $z_{k0} \neq 0,\; k = 1,\ldots,n$ *and* $\beta > 0,\; \alpha_i > 0,\; i = 1,\ldots,n$. *Then the function of* S_μ,

$$S_\mu^{-\beta} \sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} (S_\mu^{-\alpha_1})^{i_1} \times \cdots \times (S_\mu^{-\alpha_n})^{i_n} = \sum_{i_1,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} h_{(\beta+\alpha_1 i_1+\cdots+\alpha_n i_n)\mu}(t),$$

where $h_\mu(t)$ *is given by* (5), *defines an element of the ring* C_{-1}.

For the proof of this theorem we refer to [15]. We give here some operational relations, which will be used in the further discussions. For more operational relations we refer to [11], [15], and [21].

For $\rho \in \mathbf{R}$ (or $\rho \in \mathbf{C}$)

$$\frac{I}{S_\mu - \rho} = t^{\mu-1} E_{\mu,\mu}(\rho t^\mu), \tag{21}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function defined by (see [10, Vol.3])

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}\,, \quad \alpha > 0,\; |z| < \infty,$$

as can formally be obtained as a geometric series:

$$\frac{I}{S_\mu - \rho} = \frac{I}{\frac{I}{h_\mu} - \rho} = \frac{h_\mu}{I - \rho h_\mu} = \sum_{k=0}^{\infty} \rho^k h_\mu^{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{\rho^k t^{(k+1)\mu-1}}{\Gamma(\mu k+\mu)} = t^{\mu-1} E_{\mu,\mu}(\rho t^\mu).$$

The m-fold convolution of the right-hand side of the relation (21) gives us the operational relation:

$$\frac{I}{(S_\mu - \rho)^m} = t^{\mu m-1} E^m_{\mu,m\mu}(\rho t^\mu),\ m \in \mathbf{N}, \tag{22}$$

where

$$E^m_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{(m)_k z^k}{k!\Gamma(\alpha k+\beta)},\ \alpha > 0,\ |z| < \infty,\ (m)_k = \prod_{i=0}^{k-1}(m+i).$$

Let $\beta > 0,\ \alpha_i > 0,\ i = 1, \ldots, n.$ We then have the operational relation

$$\frac{S_\mu^{-\beta}}{I - \sum_{i=1}^n \lambda_i S_\mu^{-\alpha_i}} = t^{\beta\mu-1} E_{(\alpha_1\mu,\ldots,\alpha_n\mu),\beta\mu}(\lambda_1 t^{\alpha_1\mu}, \ldots, \lambda_n t^{\alpha_n\mu}) \tag{23}$$

with the multivariate Mittag-Leffler function

$$E_{(a_1,\ldots,a_n),b}(z_1, \ldots, z_n) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} (k; l_1, \ldots, l_n) \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)}$$

and the multinomial coefficients

$$(k; l_1, \ldots, l_n) := \frac{k!}{l_1! \times \cdots \times l_n!}\ .$$

In the case of Riemann-Liouville fractional derivative (1) the same scheme for developing an operational calculus can be used. We only give some results which are specific for this case.

Definition 2.4. A function $y \in C_{-1}$ is said to be in the space $\Omega_\mu(C_{-1})$ with $\mu \geq 0$, if $D^\mu y \in C_{-1}$.

The properties of this space of functions were given in [15], [21]. In particular, the space $\Omega_\mu(C_{-1})$ contains the functions $y \in C_{-1}$ which are representable in the form

$$y(t) = (J^\mu g)(t),\ g \in C_{-1}.$$

For such functions the Riemann-Liouville fractional derivative is not only a *left* inverse of the Riemann-Liouville fractional integral but also a *right* inverse one. In the case of the *whole* space $\Omega_\mu(C_{-1})$, we have the following theorem.

Theorem 2.9. *If* $y \in \Omega_\mu(C_{-1}),\ m-1 < \mu \le m \in \mathbf{N}$, *then*

$$(Fy)(t) := ((\mathrm{Id} - J^\mu D^\mu)y)(t) = \sum_{k=1}^{m} \frac{t^{\mu-k}}{\Gamma(\mu-k+1)} \lim_{t\to 0+} (D^{\mu-k}y)(t), \tag{24}$$

where the operator

$$F := \mathrm{Id} - J^\mu D^\mu$$

is called a projector of the operator J^μ, *and* Id *is an identity operator on the space* $\Omega_\mu(C_{-1})$.

Using formula (24) we arrive at the following representation of the Riemann-Liouville fractional derivative.

Theorem 2.10. *Let* $y \in \Omega(C_{-1}),\ m-1 < \mu \le m,\ m \in \mathbf{N}$. *Then,*

$$D^\mu y = S_\mu \cdot y - S_\mu \cdot \tilde{y}_\mu, \quad \tilde{y}_\mu(t) := \sum_{k=1}^{m} \frac{t^{\mu-k}}{\Gamma(\mu-k+1)} \lim_{t\to 0+} (D^{\mu-k}y)(t) \tag{25}$$

in the field $\mathcal{M}_{-1}$ *of convolution quotients.*

Now we give some elements of the operational calculus for the more general multiple Erdélyi-Kober fractional derivatives.

Definition 2.5. Let $\mu > 0,\ a_i > 0,\ \alpha_i \in \mathbf{R},\ 1 \le i \le n$. Then the multiple Erdélyi-Kober fractional integrals are given by

$$\begin{aligned}(L_\mu y)(t) &:= t^\mu \left(I_{1/a_n}^{-\alpha_n, a_n\mu} \left(I_{1/a_{n-1}}^{-\alpha_{n-1}, a_{n-1}\mu} \dots \left(I_{1/a_1}^{-\alpha_1, a_1\mu} y \right)\right)\right)(t) \qquad (26)\\ &= t^\mu \left(\prod_{i=1}^{n} I_{1/a_i}^{-\alpha_i, a_i\mu} y \right)(t)\\ &= t^\mu \int_0^1 \dots \int_0^1 y(t \prod_{i=1}^{n} v_i^{a_i}) \prod_{i=1}^{n} \frac{(1-v_i)^{a_i\mu-1} v_i^{-\alpha_i}}{\Gamma(a_i\mu)} dv,\end{aligned}$$

where $dv = dv_1 \dots dv_n$ and

$$(I_\beta^{\gamma,\delta} y)(t) = \int_0^1 \frac{(1-v)^{\delta-1} v^\gamma}{\Gamma(\delta)} y(tv^{1/\beta}) dv \tag{27}$$

is an Erdélyi-Kober (E-K) fractional integral.

Remark 2.4. *For a similar but more general definition and treatise of the "multiple Erdélyi-Kober (E-K) fractional integrals and derivatives", see Kiryakova*

[18]. There, under the same notion, a multiple E-K fractional integral, a commutative composition of E-K integrals (27) *is understood:*

$$(\mathcal{I}y)(t) = t^{\beta_0}(I^{(\gamma_i),(\delta_i)}_{(\beta_i),n}y)(t) := t^{\beta_0}\left(\prod_{i=1}^{n} I^{\gamma_i,\delta_i}_{\beta_i}y\right)(t)$$

$$= t^{\beta_0}\int_0^1 H^{n,0}_{n,n}\left[\sigma\left|\begin{array}{c}(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i})\\ (\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i})\end{array}\right.\right] y(t\sigma)d\sigma\ ,$$

where $\gamma_i \in \mathbf{R}, \delta_i \geq 0, \beta_i > 0,\ i = 1,\dots,n, \beta_0 > 0$ *and* $\delta_i\beta_i > 0,\ i = 1,\dots,n$ *can be different, not obligatory all equal to* $\beta_0 := \mu$, *as it is in* (26): $(a_i\mu).(1/a_i) = \mu, i = 1,\dots,n$. *Note that he above multiple E-K fractional integrals can be represented (along with repeated integrals like in* (26)) *also by means of single integrals involving* H*-functions as kernels. In the case considered here, the kernel is a* $G^{n,0}_{n,n}$*-function (for the G-functions see e.g. [10, Vol.1]).*

THEOREM 2.11. *Let* $\alpha = \max\limits_{1\leq i\leq n}\left\{\frac{\alpha_i-1}{a_i}\right\}$. *Then the multiple Erdélyi-Kober fractional integrals* (26) *are linear maps of the space* C_α *into itself,*

$$L_\mu : C_\alpha \to C_{\alpha+\mu} \subset C_\alpha. \tag{28}$$

We list here some important particular cases of the multiple Erdélyi-Kober fractional integrals (26).

a) Let in formula (26), $n = 1,\ a_1 = 1,\ \alpha_1 = 0$. Then

$$(L_\mu y)(t) \equiv (J^\mu y)(t) \tag{29}$$

is the Riemann-Liouville fractional integral.

b) Let in formula (26), $a_i = 1/\beta,\ \alpha_i = -\gamma_i,\ 1 \leq i \leq n,\ \mu = \beta$. Then

$$(L_\mu y)(t) \equiv (\hat{B}y)(t) := t^\beta \int_0^1 \dots \int_0^1 y(t\prod_{i=1}^{n} u_i^{1/\beta})\prod_{i=1}^{n} u_i^{\gamma_i}\, du_1 \dots du_n \tag{30}$$

is the hyper-Bessel integral operator ([7], [8], [18, Ch.3]).

DEFINITION 2.6. Let $\mu > 0,\ a_i > 0,\ \alpha_i \in \mathbf{R},\ 1 \leq i \leq n$. The multiple Erdélyi-Kober fractional derivatives are given by

$$(D_\mu y)(t) := t^{-\mu}\prod_{i=1}^{n}\prod_{k=1}^{\eta_i}\left(k-\alpha_i-a_i\mu+a_i t\frac{d}{dt}\right)\left(\prod_{i=1}^{n} I^{-\alpha_i,\eta_i-a_i\mu}_{1/a_i}y\right)(t), \tag{31}$$

where

$$\eta_i = \begin{cases}[a_i\mu]+1,\ a_i\mu \notin \mathbf{N},\\ a_i\mu,\ a_i\mu \in \mathbf{N}.\end{cases}$$

Important particular cases of these operators are:

a) The hyper-Bessel differential operator ([7], [8], [18, Ch.3]):

$$(D_\mu y)(t) \equiv (By)(t) := t^{-\beta} \prod_{i=1}^{n} \left(\gamma_i + \frac{1}{\beta} t \frac{d}{dt} \right) y(t) \tag{32}$$

in the case $a_i = 1/\beta,\ \alpha_i = -\gamma_i,\ \eta_i = 1,\ 1 \le i \le n,\ \mu = \beta.$

b) The Riemann-Liouville fractional derivative

$$(D_\mu y)(t) \equiv (D^\mu y)(t) \tag{33}$$

in the case $n = 1,\ a_1 = 1,\ \alpha_1 = 0$, and

$$\eta_1 = \eta = \begin{cases} [\mu] + 1,\ \mu \notin \mathbf{N}, \\ \mu,\ \mu \in \mathbf{N}. \end{cases}$$

The relationship between the multiple Erdélyi-Kober fractional integrals (26) and derivatives (31) is given by the following theorem.

THEOREM 2.12. *For $y \in C_\alpha$ with $\alpha = \max\limits_{1\le i\le n} \left\{ \frac{\alpha_i - 1}{a_i} \right\}$, let $g(t) = (L_\mu y)(t)$. Then,*

$$(D_\mu L_\mu y)(t) = (D_\mu g)(t) = y(t), \tag{34}$$

that is, the operator L_μ is a right inverse of the operator D_μ.

To deal with the multiple Erdélyi-Kober fractional derivative we introduce a suitable space of functions.

DEFINITION 2.7. Denote by $\Omega_\mu^m(C_\alpha),\ m \in \mathbf{N},\ \mu > 0$, the space of all functions y, such that $D_\mu^k y \in C_\alpha,\ k = 0, \ldots, m$, where D_μ^k means the composition of k multiple Erdélyi-Kober fractional derivatives (31) for $k = 1, 2 \ldots$ and D_μ^0 is an identity operator on the space C_α.

It turns out that the operator L_μ is a left inverse of the operator D_μ on the subspace of $\Omega_\mu^1(C_\alpha)$ which consists of the functions $y \in \Omega_\mu^1(C_\alpha)$, which can be represented in the form $y(t) = (L_\mu g)(t),\ g \in C_\alpha$. This property is not valid for the whole space $\Omega_\mu^1(C_\alpha)$. In this case we have the following result.

THEOREM 2.13. *Let $y \in \Omega_\mu^1(C_\alpha)$ and*

$$\frac{\mu a_i - \eta_i + \alpha_i}{a_i} > \alpha,\ i = 1, \ldots, n. \tag{35}$$

Then,

$$(Fy)(t) = ((\mathrm{Id} - L_\mu D_\mu) y)(t) = \sum_{i=1}^{n} \sum_{k=1}^{\eta_i} C_{ik} \left[\lim_{t \to 0+} (A_{ik} y)(t) \right] t^{\mu - \frac{k - \alpha_i}{a_i}}, \tag{36}$$

where

$$C_{ik} = \frac{\prod_{j=i+1}^{n} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)) \prod_{j=1}^{i-1} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)+\eta_j)}{\prod_{j=1}^{n} \Gamma(1-\alpha_j - \frac{a_j}{a_i}(k-\alpha_i)+a_j\mu)},$$

$$(A_{ik}y)(t) = t^{-\mu+\frac{k-\alpha_i}{a_i}} \prod_{j=1}^{\eta_i - k} \left(k+j-\alpha_i - a_i\mu + a_i t\frac{d}{dt}\right)$$

$$\times \prod_{l=i+1}^{n} \prod_{j=1}^{\eta_i} \left(j-\alpha_l - a_l\mu + a_l t\frac{d}{dt}\right) \left(\prod_{j=1}^{n} I_{1/a_j}^{-\alpha_j, \eta_j - a_j\mu} y\right)(t),$$

where

$$\eta_i = \begin{cases} [a_i\mu]+1, a_i\mu \notin \mathbf{N}, \\ a_i\mu, \ a_i\mu \in \mathbf{N}, \end{cases}$$

the operator $F = \mathrm{Id} - L_\mu D_\mu$ *is the projector of the operator* L_μ *and* Id *is an identity operator on the space* $\Omega_\mu^1(C_\alpha)$.

For the Riemann-Liouville fractional integral and derivative the representation (36) coincides with (24).

In the case of hyper-Bessel integral (30) and differential (32) operators and under the conditions $\gamma_1 < \gamma_2 < \ldots < \gamma_n < \gamma_1 + 1$, we get

$$(Fy)(t) = ((\mathrm{Id} - \hat{B}B)y)(t) = \sum_{i=1}^{n} t^{-\beta\gamma_i} \beta^{i-n} \prod_{j=i+1}^{n} (\gamma_j - \gamma_i)^{-1} \tag{37}$$

$$\times \lim_{t\to 0+} \left[t^{\beta\gamma_i} \prod_{j=i+1}^{n} (\beta\gamma_j + t\frac{d}{dt}) y(t) \right],$$

as found in [18, p.111].

To develop an operational calculus of Mikusiński type for the multiple Erdélyi-Kober fractional derivative we introduce a family of convolutions for this operator.

Theorem 2.14. *Let the following conditions hold*

$$\alpha = \max_{1\le i\le n} \left\{ \frac{\alpha_i - 1}{a_i} \right\}, \ \lambda \ge \max_{1\le i\le n} \left\{ \frac{1-\alpha_i}{a_i} \right\}. \tag{38}$$

Then the operation

$$(f \overset{\lambda}{*} g)(t) = t^\lambda \left(\prod_{i=1}^{n} I_{1/a_i}^{1-2\alpha_i, \alpha_i + a_i\lambda - 1} (f \circ g) \right)(t), \tag{39}$$

$$(f\circ g)(t) = \int_0^1 \ldots \int_0^1 f(t\prod_{i=1}^{n} u_i^{a_i}) g(t\prod_{i=1}^{n}(1-u_i)^{a_i}) \prod_{i=1}^{n} (u_i(1-u_i))^{-\alpha_i}\, du_1 \ldots du_n$$

is a convolution without divisors of zero of the multiple Erdélyi-Kober fractional integral L_μ *(26) on the space* C_α *in the following sense (cf. [9]):*

$$((L_\mu f) \overset{\lambda}{*} g)(t) = (L_\mu (f \overset{\lambda}{*} g))(t) \quad (\forall f, g \in C_\alpha). \tag{40}$$

As in the case of the operational calculus for the Caputo fractional derivative we get the following result.

Theorem 2.15. *The space* C_α *with the operations* $\overset{\lambda}{*}$ *and the ordinary addition becomes a commutative ring* $(C_\alpha, \overset{\lambda}{*}, +)$ *without divisors of zero.*

A multiple Erdélyi-Kober fractional integral can be represented in this ring as an operator of multiplication.

Theorem 2.16. *Let conditions* (38) *hold true and*

$$\lambda < \mu - \alpha. \tag{41}$$

Then the multiple Erdélyi-Kober fractional integral (26) *has the convolution representation*

$$(L_\mu y)(t) = (y \overset{\lambda}{*} h)(t),\ \ h(t) = \frac{t^{\mu-\lambda}}{\prod_{i=1}^n \Gamma(1-\alpha_i + a_i(\mu-\lambda))}\ . \tag{42}$$

In the further discussions we shall assume, that conditions (38) and (41) hold true anywhere.

Remark 2.5. *Using the relation* (42) *and direct evaluations, we get*

$$(L_\mu^n y)(t) = (L_{n\mu} y)(t) = (y \overset{\lambda}{*} h^n)(t), \tag{43}$$

where

$$h^n(t) := \frac{t^{n\mu-\lambda}}{\prod_{i=1}^n \Gamma(1-\alpha_i + a_i(n\mu-\lambda))}. \tag{44}$$

In order to get similar representation for the multiple Erdélyi-Kober fractional derivatives, we should extend the ring C_α to the field $\mathcal{M}_{\alpha,\lambda}$ of convolution quotients by factorizing the set $C_\alpha \times (C_\alpha - \{0\})$ with respect to the equivalence relation

$$(f,g) \sim (f_1,g_1) \Leftrightarrow (f \overset{\lambda}{*} g_1)(t) = (g \overset{\lambda}{*} f_1)(t). \tag{45}$$

The operations of addition and multiplication are defined as usual (see the case of the Caputo fractional derivative).

Definition 2.8. The algebraic inverse of a multiple Erdélyi-Kober fractional integral L_μ is said to be the element S of the field $\mathcal{M}_{\alpha,\lambda}$, which is reciprocal to the element h defined by (42) in the field $\mathcal{M}_{\alpha,\lambda}$, that is,

$$S = I/h = h/(h \overset{\lambda}{*} h) = h/h^2, \tag{46}$$

where $I = \frac{h}{h}$ denotes the identity element of the field $\mathcal{M}_{\alpha,\lambda}$ with respect to the operation of multiplication.

Then we get the following result.

THEOREM 2.17. *Let $y \in \Omega_\mu^m(C_\alpha)$. Then the relation*

$$D_\mu^m y = S^m \cdot y - \sum_{k=0}^{m-1} S^{m-k} \cdot y_k, \quad y_k(t) := (FD_\mu^k y)(t), \tag{47}$$

holds true in the convolution quotient field $\mathcal{M}_{\alpha,\lambda}$, where $F = \mathrm{Id} - L_\mu D_\mu$ is the projector (36) *of the operator L_μ.*

For the applications of the developed operation calculus, representations of some elements of the convolution quotient field by elements of the initial ring of functions are important.

We can obtain a class of such functions for our operational calculus using the following theorem.

THEOREM 2.18. *Let the power series of complex variable z with complex coefficients be convergent at the point $z_0 \neq 0$, that is,*

$$\sum_{k=0}^{\infty} b_k z_0^k = A \in \mathbf{C}.$$

Then,

$$\sum_{k=1}^{\infty} b_k S^{-k} = \sum_{k=1}^{\infty} b_k h^k(t), \; h^k(t) = \frac{t^{k\mu-\lambda}}{\prod_{i=1}^n \Gamma(1-\alpha_i + a_i(k\mu - \lambda))} \tag{48}$$

is an element of the ring C_α.

It follows from Theorem 2.18, representation (43), and the formula for the sum of infinite geometric progression that ($\rho \in \mathbf{R}$ or $\rho \in \mathbf{C}$)

$$\frac{I}{S-\rho} = \frac{h}{I-\rho h} = h(I + \rho h + \rho^2 h^2 + \ldots) = t^{\mu-\lambda} \tag{49}$$

$$\times \sum_{k=0}^{\infty} \frac{(\rho t^\mu)^k}{\prod_{i=1}^n \Gamma(1-\alpha_i + a_i(\mu-\lambda) + a_i \mu k)} = t^{\mu-\lambda} E\left((1-\alpha_i + a_i(\mu-\lambda), a_i\mu)_n; \rho t^\mu\right),$$

where

$$E\left((\alpha,\beta)_n; z\right) := E\left((\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n); z\right) := \sum_{k=0}^{\infty} \frac{z^k}{\prod_{i=1}^n \Gamma(\alpha_i + \beta_i k)}, \; \sum_{i=1}^n \alpha_i > 0$$

is a Mittag-Leffler function of vector index.

By direct evaluations we have from relation (49),

$$\begin{aligned} \frac{I}{(S-\rho)^m} &= t^{\mu m-\lambda} \sum_{k=0}^{\infty} \frac{(m)_k (\rho t^\mu)^k}{k! \prod_{i=1}^n \Gamma(1-\alpha_i + a_i(\mu m-\lambda) + a_i\mu k)} \\ &= t^{\mu m-\lambda} E_m\left((1-\alpha_i + a_i(\mu m-\lambda), a_i\mu)_n; \rho t^\mu\right), \end{aligned} \tag{50}$$

where

$$E_m\left((\alpha,\beta)_n;z\right) := E_m\left((\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n);z\right) := \sum_{k=0}^{\infty}\frac{(m)_k z^k}{k!\prod_{i=1}^{n}\Gamma(\alpha_i+\beta_i k)}$$

with the condition $\sum_{i=1}^{n}\alpha_i > 0$.

Relation (49) is reduced in the case of the Caputo (or the Riemann-Liouville) fractional derivative to the following one under the condition $1 \le \lambda < 1+\mu$ (compare with (21)):

$$\frac{I}{S-\rho} = t^{\mu-\lambda}\sum_{k=0}^{\infty}\frac{(\rho t^{\mu})^k}{\Gamma(1+\mu-\lambda+\mu k)} = t^{\mu-\lambda}E_{1+\mu-\lambda,\mu}(\rho t^{\mu}), \tag{51}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function.

In the case of the hyper-Bessel differential operator (32), we obtain from relation (49) under conditions $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_n$ and $\beta(1+\gamma_n) \le \lambda < \beta(\gamma_1+2)$:

$$\frac{I}{S-\rho} = \frac{t^{\beta-\lambda}}{\prod_{i=1}^{n}\Gamma(2+\gamma_i-\lambda/\beta)}\,{}_1F_n(1;2+\gamma_1-\lambda/\beta,\ldots,2+\gamma_n-\lambda/\beta;\rho t^{\beta}) \tag{52}$$

and we have for $\lambda = \beta(\gamma_n+1)$,

$$\frac{I}{S-\rho} = \frac{t^{-\beta\gamma_n}}{\prod_{i=1}^{n}\Gamma(1+\gamma_i-\gamma_n)}\,{}_0F_{n-1}(1+\gamma_1-\gamma_n,\ldots,1+\gamma_{n-1}-\gamma_n;\rho t^{\beta}), \tag{53}$$

where ${}_0F_{n-1}(z)$ is the so-called hyper-Bessel function of Delerue (see e.g. [18, eq.(D.4), p.337]).

From relation (50) in this case it follows that

$$\frac{I}{(S-\rho)^m} = \frac{t^{\beta m-\lambda}}{\prod_{i=1}^{n}\Gamma(1+m+\gamma_i-\lambda/\beta)} \tag{54}$$

$$\times {}_1F_n(m;1+m+\gamma_1-\lambda/\beta,\ldots,1+m+\gamma_n-\lambda/\beta;\rho t^{\beta}),$$

where ${}_1F_n(z)$ is a generalized hypergeometric function.

Remark 2.6. *We can represent the right-hand side of the relation* (50) *in terms of the Fox H-function:*

$$\frac{I}{(S-\rho)^m} = \frac{t^{\mu m-\alpha}}{(m-1)!}H^{1,1}_{n+1,1}\left(-\rho t^{\mu}\left|\begin{matrix}(1,1),(1-\alpha_i+a_i(\mu m-\lambda),a_i\mu)_{1,n}\\(m,1)\end{matrix}\right.\right). \tag{55}$$

Remark 2.7. *An operational calculus for the multiple Erdélyi-Kober fractional derivative can be developed also on the basis of the generalized Obrechkoff transform with the power weight*

$$(\mathcal{O}_{\beta}y)(t) = t^{\beta}\int_0^{\infty}H^{0,n}_{n,0}\left(\frac{t}{u}\left|\begin{matrix}(\alpha,a)_{1,n}\\-\end{matrix}\right.\right)y(u)\frac{du}{u} \tag{56}$$

which plays in this calculus the same role as the Laplace transform in the Mikusiński operational calculus. Let us note that the generalized Obrechkoff transform can be also represented in the form

$$(\mathcal{O}_\beta y)(t) = t^\beta \int_0^\infty \Phi_n \left(t/u \mid (\alpha_i, a_i)_1^n\right) y(u) \frac{du}{u}\,, \tag{57}$$

where

$$\Phi(\tau|(\alpha_i, a_i)_1^n) = \frac{\tau^{(\alpha_n-1)/a_n}}{a_n} \int_0^\infty \cdots \int_0^\infty \exp\left\{-\sum_{i=1}^{n-1} u_i - \tau^{-\frac{1}{a_n}} \prod_{i=1}^{n-1} u_i^{-a_i/a_n}\right\}$$
$$\times \prod_{i=1}^{n-1} u_i^{-a_i \frac{1-\alpha_n}{a_n} - \alpha_i}\, du_1 \dots du_{n-1}.$$

Remark 2.8. *In the special case of hyper-Bessel differential operator* (32) *transformation* (57) *is the Obrechkoff transform, proposed in [8] as a transform basis for an operational calculus for such an operator. More details on the Obrechkoff transform, its representations and operational properties related to the theory of hyper-Bessel operators, can be found in Dimovski [8] and Kiryakova [18, Ch.3]. A modification of the generalized Obrechkoff transform is considered also by Kiryakova [19] in close relation to the multiple Mittag-Leffler functions* $E((\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n))$ *referred to in this section as Mittag-Leffler functions of vector index.*

3. Fractional differential equations

In this section we apply the developed operational calculi for solving fractional differential equations. Special emphasis will be given to the equations with the Caputo fractional derivatives. As to the fractional differential equations with the Riemann-Liouville fractional derivatives and with the multiple Erdélyi-Kober fractional derivatives the same scheme can be used there. These cases were considered in [15], [21] and [1], [22], respectively.

We first consider some simple fractional differential equations, which were already studied by using the Laplace transform technique (see [13] and references there). We begin with the initial value problem ($\mu > 0$)

$$\begin{cases} (D_*^\mu y)(t) - \lambda y(t) = g(t), \\ y^{(k)}(0) = c_k \in \mathbf{R}, \quad k = 0, \dots, m-1,\ m-1 < \mu \le m,\ \lambda \in \mathbf{R}. \end{cases} \tag{58}$$

The function g is assumed to lie in C_{-1} if $\mu \in \mathbf{N}$, in C_{-1}^1 if $\mu \notin \mathbf{N}$, and the unknown function y is to be determined in the space C_{-1}^m.

Making use of the relation (14), the initial value problem (58) can be reduced to the following algebraic equation in the field $\mathcal{M}_{-1}$ of convolution quotients:

$$S_\mu \cdot y - \lambda y = S_\mu \cdot y_\mu + g, \quad y_\mu(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!}, \quad m-1 < \mu \le m,$$

whose unique solution in the field $\mathcal{M}_{-1}$ has the form:

$$y = y_g + y_h = \frac{I}{S_\mu - \lambda} \cdot g + \frac{S_\mu}{S_\mu - \lambda} \cdot y_\mu.$$

It turns out that the right-hand side of this relation can be interpreted as a function from the space C_{-1}^m, that is, as a classical solution of the initial value problem (58).

It follows from the operational relation (21) and the embedding of the ring C_{-1} into the field $\mathcal{M}_{-1}$, that the first term of this relation, y_g (solution of the inhomogeneous fractional differential equation (58) with zero initial conditions), is represented in the form

$$y_g(t) = \int_0^t \tau^{\mu-1} E_{\mu,\mu}(\lambda \tau^\mu) g(t-\tau)\, d\tau. \tag{59}$$

As to the second term, y_h, it is a solution of the homogeneous fractional differential equation (58) with the given initial conditions and we have

$$y_h(t) = \sum_{k=0}^{m-1} c_k u_k(t),\ u_k(t) = \frac{S_\mu}{S_\mu - \lambda} \cdot \{\frac{t^k}{k!}\}. \tag{60}$$

Making use of the relation

$$\frac{t^k}{k!} = h_{k+1}(t) = h^\mu_{(k+1)/\mu}(t) = \frac{I}{S_\mu^{(k+1)/\mu}}\ , \tag{61}$$

the formula (19), and the operational relation (23), we get the representation of the functions $u_k(t),\ k = 0, \dots, m-1$ in terms of the generalized Mittag-Leffler function:

$$u_k(t) = \frac{S_\mu}{S_\mu - \lambda} \cdot \{\frac{t^k}{k!}\} = \frac{S_\mu^{-(k+1)/\mu}}{I - \lambda S_\mu^{-1}} = t^k E_{\mu,k+1}(\lambda t^\mu).$$

Furthermore, due to representation (5) of the Riemann-Liouville fractional integral, we have

$$u_k(t) = (J^k u_0)(t),\ u_0(t) = E_{\mu,1}(\lambda t^\mu) := E_\mu(\lambda t^\mu),$$

where $E_\alpha(z) = \sum_{k=0}^\infty z^k/\Gamma(\alpha k+1)$ is the classical Mittag-Leffler function (see [10, Vol.3]). Using the last representation, we arrive at the relations

$$u_k^{(l)}(0) = \delta_{kl},\ k, l = 0, \dots, m-1,$$

and therefore the m functions $u_k(t),\ k = 0, \dots, m-1$ represent the general solution of the homogeneous fractional differential equation (58). Summarizing the obtained results, we get the solution of the initial value problem (58) in the form:

$$y(t) = \int_0^t \tau^{\mu-1} E_{\mu,\mu}(\lambda\tau^\mu) g(t-\tau)\, d\tau + \sum_{k=0}^{m-1} c_k t^k E_{\mu,k+1}(\lambda t^\mu),$$

which can be rewritten in the case $\lambda \neq 0$ in terms of the Mittag-Leffler function:

$$y(t) = \frac{1}{\lambda} \int_0^t \frac{d}{d\tau} \left(E_\mu(\lambda\tau^\mu)\right) g(t-\tau)\, d\tau + \sum_{k=0}^{m-1} c_k (J^k E_\mu(\lambda\tau^\mu))(t).$$

The next equation,

$$\begin{cases} y'(t) - \lambda_1 (D_*^\mu y)(t) - \lambda_2 y(t) = g(t), \\ y(0) = c_0 \in \mathbf{R}, \quad 0 < \mu < 1, \ \lambda_1, \lambda_2 \in \mathbf{R} \end{cases} \tag{62}$$

with $\mu = 1/2$ and $\lambda_1 < 0,\ \lambda_2 < 0$ corresponds to the Basset problem, a classical problem in fluid dynamics (see [13], [24]). We treat the general problem (62). The function $g \in C_{-1}$ is given, and the unknown function y is to be determined in the space C_{-1}^1.

With the help of relation (14) the problem under consideration can be reduced to the algebraic equation in the field $\mathcal{M}_{-1}$:

$$S_1 \cdot y - \lambda_1 S_\mu \cdot y - \lambda_2 y = g + S_1 \cdot y_1 - \lambda_1 S_\mu \cdot y_\mu,\ y_1(t) \equiv y_\mu(t) \equiv c_0.$$

Applying the relation (20) we represent a unique solution of this equation in the field $\mathcal{M}_{-1}$ in the form:

$$y = y_g + y_h = \frac{I}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot g + \frac{S_1 - \lambda_1 S_1^\mu}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot y_1, \qquad y_1(t) \equiv c_0.$$

Using now the relations (19) and (23) we arrive at the representation

$$\frac{I}{S_1 - \lambda_1 S_1^\mu - \lambda_2} = \frac{S_1^{-1}}{I - \lambda_1 S_1^{-(1-\mu)} - \lambda_2 S_1^{-1}} = E_{(1-\mu,1),1}(\lambda_1 t^{1-\mu}, \lambda_2 t)$$

with the multivariate Mittag-Leffler function. We also have, using the same technique and (11), (61), the relation

$$y_h(t) = \frac{S_1 - \lambda_1 S_1^\mu}{S_1 - \lambda_1 S_1^\mu - \lambda_2} \cdot \{c_0\} = \left[I + \frac{\lambda_2}{S_1 - \lambda_1 S_1^\mu - \lambda_2}\right] \cdot \frac{c_0 I}{S_1}$$

$$= c_0 \left[\frac{I}{S_1} + \lambda_2 \frac{S_1^{-2}}{I - \lambda_1 S_1^{\mu-1} - \lambda_2 S_1^{-1}}\right] = c_0 \left[1 + \lambda_2 t E_{(1-\mu,1),2}(\lambda_1 t^{1-\mu}, \lambda_2 t)\right].$$

The unique solution of the initial value problem (62) has then the form

$$y(t) = \int_0^t E_{(1-\mu,1),1}(\lambda_1 \tau^{1-\mu}, \lambda_2 \tau) g(t-\tau)\, d\tau \; + \; c_0 \left[1 + \lambda_2 t E_{(1-\mu,1),2}(\lambda_1 t^{1-\mu}, \lambda_2 t)\right].$$

We consider now the general n-term linear differential equation with the constant coefficients and the Caputo fractional derivatives.

THEOREM 3.1. *Let $\mu > \mu_1 > \ldots > \mu_n \geq 0,\ m_i - 1 < \mu_i \leq m_i,\ m_i \in \mathbf{N}_0 = \mathbf{N} \cup \{0\},\ \lambda_i \in \mathbf{R},\ i = 1, \ldots, n$. The initial value problem*

$$\begin{cases} (D_*^{\mu} y)(t) - \sum_{i=1}^{n} \lambda_i (D_*^{\mu_i} y)(t) = g(t), \\ y^{(k)}(0) = c_k \in \mathbf{R}, \quad k = 0, \ldots, m-1,\ m-1 < \mu \leq m, \end{cases} \tag{63}$$

where the function g is assumed to lie in C_{-1}, if $\mu \in \mathbf{N}$, in C^1_{-1} if $\mu \notin \mathbf{N}$, and the unknown function y is to be determined in the space C^m_{-1}, has a solution, unique in the space C^m_{-1}, of the form:

$$y(t) = y_g(t) + \sum_{k=0}^{m-1} c_k u_k(t),\ x \geq 0. \tag{64}$$

Here

$$y_g(t) = \int_0^t \tau^{\mu-1} E_{(\cdot),\mu}(\tau) g(t-\tau)\, d\tau \tag{65}$$

is a solution of problem (63) *with zero initial conditions, and the system of functions*

$$u_k(t) = \frac{t^k}{k!} + \sum_{i=l_k+1}^{n} \lambda_i t^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(t), \quad k = 0, \ldots, m-1 \tag{66}$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl},\ k, l = 0, \ldots, m-1$. The function

$$E_{(\cdot),\beta}(t) = E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\beta}(\lambda_1 t^{\mu-\mu_1}, \ldots, \lambda_n t^{\mu-\mu_n}) \tag{67}$$

is a particular case of the multivariate Mittag-Leffler function (23) *and the natural numbers $l_k,\ k = 0, \ldots, m-1$ are determined from the condition*

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases} \tag{68}$$

In the case $m_i \leq k,\ i = 0, \ldots, m-1$ we set $l_k := 0$, and if $m_i \geq k+1,\ i = 0, \ldots, m-1$, then $l_k := n$.

REMARK 3.1. *The results of Theorem 3.1 can be used in some cases for the initial value problem (63) with the Riemann-Liouville fractional derivatives instead of the Caputo fractional derivatives. In particular, as we have seen in Remark 2.3, $(D^{\mu} y)(t) \equiv (D_*^{\mu} y)(t)$, if $\mu = m \in \mathbf{N}$ or $y^{(k)}(0) = 0,\ k = 0, \ldots, m-1,\ m-1 < \mu \leq m$. In the case $0 < \mu < 1$ we can use the relation*

$$(D^\mu y)(t) = (D_*^\mu y)(t) + \frac{y(0+)}{\Gamma(1-\mu)} t^{-\mu}$$

to reduce the initial value problem with the Riemann-Liouville fractional derivatives to the initial value problem of the type (63).

The solution (64)-(66) of the initial value problem (63) was obtained in terms of the Mittag-Leffler type function $E_{(\cdot),\beta}(t)$, which is given by its series representation:

$$E_{(\cdot),\beta}(t) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} (k; l_1, \ldots, l_n) \frac{\prod_{i=1}^n (\lambda_i t^{\mu-\mu_i})^{l_i}}{\Gamma(\beta + \sum_{i=1}^n (\mu-\mu_i) l_i)} \,. \tag{69}$$

Let us find an integral representation of this function. We shall use the Hankel integral representation of the Gamma-function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{Ha(0+)} e^\zeta \zeta^{-z} \, d\zeta, \;\; z \in \mathbf{C},$$

where $Ha(\epsilon+)$ is the Hankel path, a loop which starts from $-\infty$ along the lower side of the negative real axis, encircles the circular disc $|\zeta| = \zeta_0 > \epsilon$ in the positive sense and ends at $-\infty$ along the upper side of the negative real axis. Then we substitute this representation into (69) and we get

$$E_{(\cdot),\beta}(t) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots,l_n\geq 0}} (k; l_1, \ldots, l_n) \prod_{i=1}^{n} (\lambda_i t^{\mu-\mu_i})^{l_i}$$
$$\times \frac{1}{2\pi i} \int_{Ha(0+)} e^\zeta \zeta^{-\beta - \sum_{i=1}^n (\mu-\mu_i) l_i} \, d\zeta.$$

Changing the order of integration and summation, using the geometric progression formula to get a closed form for a sum in integrand and substituting $\zeta = st$, we finally get

$$E_{(\cdot),\beta}(t) = t^{1-\beta} \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{st} s^{\mu-\beta} \, ds}{s^\mu - \sum_{i=1}^n \lambda_i s^{\mu_i}}, \;\; \lambda = \max\left\{1, \left(\sum_{i=1}^n |\lambda_i|\right)^{1/(\mu-\mu_1)}\right\}. \tag{70}$$

Remark 3.2. *Applying the representation* (70) *to the formulas* (65) *and* (66), *we rewrite the solution* (64) *of the initial value problem* (63) *in the form*

$$y(t) = \int_0^t u_\delta(\tau) g(t-\tau) + \sum_{k=0}^{m-1} c_k u_k(t), \quad t \geq 0, \tag{71}$$

where

$$u_\delta(t) = \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{st} \, ds}{s^\mu - \sum_{i=1}^n \lambda_i s^{\mu_i}} \,, \tag{72}$$

$$u_k(t) = \frac{t^k}{k!} + \frac{1}{2\pi i}\int_{Ha(\lambda+)} \frac{e^{st}\sum_{i=l_k+1}^{n}\lambda_i s^{\mu_i}}{s^{\mu} - \sum_{i=1}^{n}\lambda_i s^{\mu_i}}\frac{ds}{s^{k+1}} \tag{73}$$

$$= \frac{1}{2\pi i}\int_{Ha(\lambda+)} \frac{e^{st}\left[s^{\mu} - \sum_{i=1}^{l_k}\lambda_i s^{\mu_i}\right]}{s^{\mu} - \sum_{i=1}^{n}\lambda_i s^{\mu_i}}\frac{ds}{s^{k+1}}, \quad k = 0, \ldots, m-1.$$

In particular, if $\mu_n = 0, \ \lambda_n \neq 0$, *then* $l_0 = n-1$ *and we have*

$$u_0(t) = \frac{1}{2\pi i}\int_{Ha(\lambda+)} \frac{e^{st}\left[s^{\mu} - \sum_{i=1}^{n-1}\lambda_i s^{\mu_i}\right]}{s^{\mu} - \sum_{i=1}^{n-1}\lambda_i s^{\mu_i} - \lambda_n}\frac{ds}{s}.$$

In this situation we have the relation

$$u_\delta(t) = \frac{1}{\lambda_n}u_0'(t).$$

If $\mu_n > 0$, *we get* $u_0(t) \equiv 1$.

Remark 3.3. *The initial value problem* (63) *for the three cases: 1)* $n = 1, \ \mu_1 = 0, \ \lambda_1 = -1$, *2)* $n = 2, \ \mu = 1, \ \lambda_2 = -1, \ \mu_2 = 0$, *and 3)* $n = 2, \ \mu = 2, \ \lambda_2 = -1, \ \mu_2 = 0$ *was considered in [13] by using the Laplace transform method. In this interesting paper the form* (71)-(73) *of the solution was obtained and used, by evaluating the contribution of poles of the integrand by the residue theorem and transforming the Hankel path* $Ha(\lambda+)$ *into the* $Ha(0+)$, *to represent it as a sum of oscillatory and monotone parts. In addition, asymptotic expansions, plots and interesting particular cases are given there. General results concerning the methods of evaluation of the poles of integrand in the integral representations of the type* (72), (73), *asymptotic expansions of such representations as well as a lot of interesting particular cases can be found in the paper [16].*

We give now some examples.

Example 1. Let the right part of the fractional differential equation (63) be a power function:

$$g(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}, \ \alpha > -1, \text{ if } \mu \in \mathbf{N}, \ \alpha \geq 0, \text{ if } \mu \notin \mathbf{N}.$$

Since

$$\frac{t^\alpha}{\Gamma(\alpha+1)} = h_{\alpha+1}(t) = h^{\mu}_{(\alpha+1)/\mu}(t) = S_\mu^{-(\alpha+1)/\mu},$$

we get by using (19), (23), and (70) the following representations of the part $y_g(t)$ of the solution (64):

$$y_g = \frac{I}{S_\mu - \sum_{i=1}^{n}\lambda_i S_\mu^{\mu_i/\mu}}\cdot g = \frac{I}{S_\mu - \sum_{i=1}^{n}\lambda_i S_\mu^{\mu_i/\mu}}\cdot\frac{I}{S_\mu^{(\alpha+1)/\mu}}$$

$$= \frac{S_\mu^{-(\mu+\alpha+1)/\mu}}{I - \sum_{i=1}^{n}\lambda_i S_\mu^{-(\mu-\mu_i)/\mu}} = t^{\mu+\alpha}E_{(\cdot),\mu+\alpha+1}(t)$$

$$= \frac{1}{2\pi i} \int_{Ha(\lambda+)} \frac{e^{st} s^{-\alpha-1}\, ds}{s^{\mu} - \sum_{i=1}^{n} \lambda_i s^{\mu_i}} \,.$$

EXAMPLE 2. We consider now the equation (63) with $\mu_i = (n-i)\alpha,\ i = 1, \dots, n,\ \mu = n\alpha,\ q-1 < \mu \le q,\ q \in \mathbf{N}$. Then the solution (64) can be represented in terms of the generalized Mittag-Leffler function $E^{m}_{\alpha,\beta}(t)$ (22). Indeed, using relation (20) and representing the corresponding rational function as a sum of partial fractions, we get

$$\begin{aligned} y_g &= \frac{I}{S_\mu - \sum_{i=1}^{n} \lambda_i S_\mu^{\mu_i/\mu}} \cdot g = \frac{I}{S_\alpha^n - \sum_{i=1}^{n} \lambda_i S_\alpha^{n-i}} \cdot g \\ &= \left[\sum_{j=1}^{p} \sum_{m=1}^{n_j} \frac{c_{jm}}{(S_\alpha - \beta_j)^m} \right] \cdot g, \quad n_1 + \cdots + n_p = n. \end{aligned}$$

Operational relation (22) gives us the representation

$$y_g(t) = \int_0^t u_\delta(\tau) g(t-\tau)\, d\tau,$$

where

$$u_\delta(t) = \sum_{j=1}^{p} \sum_{m=1}^{n_j} c_{jm} t^{\alpha m - 1} E^{m}_{\alpha, m\alpha}(\beta_j t^\alpha).$$

We have also ($k = 0, \dots, q-1$)

$$y_h(t) = \sum_{k=0}^{q-1} c_k u_k(t),\ u_k(t) = \frac{t^k}{k!} + \left\{ \frac{t^k}{k!} \right\} \cdot \frac{\sum_{i=l_k+1}^{n} \lambda_i S_\alpha^{n-i}}{S_\alpha^n - \sum_{i=1}^{n} \lambda_i S_\alpha^{n-i}} = \frac{t^k}{k!} + (J^{k+1} v_k)(t),$$

$$\begin{aligned} v_k(t) &= \frac{\sum_{i=l_k+1}^{n} \lambda_i S_\alpha^{n-i}}{S_\alpha^n - \sum_{i=1}^{n} \lambda_i S_\alpha^{n-i}} = \sum_{j=1}^{p_k} \sum_{m=1}^{n_{jk}} \frac{c_{jmk}}{(S_\alpha - \beta_{jk})^m} \\ &= \sum_{j=1}^{p_k} \sum_{m=1}^{n_{jk}} c_{jmk} t^{\alpha m - 1} E^{m}_{\alpha, m\alpha}(\beta_{jk} t^\alpha), \qquad \sum_{j=1}^{p_k} n_{jk} = n. \end{aligned}$$

In the case $\alpha \in \mathbf{Q}$ the generalized Mittag-Leffler function $E^{m}_{\alpha,m\alpha}(x)$ can be represented in terms of the special functions of hypergeometric type (see [13]).

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